# Početní část 2 - 18.1.2021

3.

$$\begin{aligned}\int \sin(2x)\sqrt{|1-2\sin^2 x|}\,dx &= 2\int \sin x\cos x\sqrt{|1-2\sin^2 x|}\,dx \\ &\overset{y=\sin x; dy=\cos x dx}{=} 2\int y\sqrt{|1-2y^2|}\,dy \\ &\overset{c}{=} -\frac{1}{3}|1-2y^2|^{\frac{3}{2}}\operatorname{sgn}(1-2y^2) \\ &= -\frac{1}{3}|1-2\sin^2 x|^{\frac{3}{2}}\operatorname{sgn}(1-2\sin^2 x) =: F(x).\end{aligned}$$

Protože integrand je spojitý na $\mathbb{R}$, očekáváme primitivní funkci na $\mathbb{R}$. Podle věty o lepení stačí ověřit, že $F$ je spojitá na $\mathbb{R}$. Platí $F = g \circ h$ pro $g(x) = -\frac{1}{3}|x|^{\frac{3}{2}}\operatorname{sgn}(x)$ a $h(x) = 1 - 2\sin^2 x$. Funkce $h$ je spojitá na $\mathbb{R}$ a totéž platí pro funkci $g$ (stačí ověřit $\lim\limits_{x\to 0\pm} h(x) = 0 = h(0)$). Tedy $F$ je spojitá na $\mathbb{R}$.

4. (a)

$$
\begin{aligned}
\sin x &= \sin\tfrac{\pi}{4} + \sin'(\tfrac{\pi}{4})(x-\tfrac{\pi}{4}) + \sin''(\tfrac{\pi}{4})\tfrac{(x-\frac{\pi}{4})^2}{2} + o((x-\tfrac{\pi}{4})^2)\\
&= \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) - \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^2}{2} + o((x-\tfrac{\pi}{4})^2)\\
\cos x &= \cos\tfrac{\pi}{4} + \cos'(\tfrac{\pi}{4})(x-\tfrac{\pi}{4}) + \cos''(\tfrac{\pi}{4})\tfrac{(x-\frac{\pi}{4})^2}{2} + o((x-\tfrac{\pi}{4})^2)\\
&= \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) - \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^2}{2} + o((x-\tfrac{\pi}{4})^2)\\
\tan x &= \tan\tfrac{\pi}{4} + \tan'(\tfrac{\pi}{4})(x-\tfrac{\pi}{4}) + o((x-\tfrac{\pi}{4})) = 1 + 2(x-\tfrac{\pi}{4}) + o((x-\tfrac{\pi}{4}))
\end{aligned}
$$

(b) Protože Taylorův polynom funkce ve jmenovateli bude mít zjevně první nenulový člen u $(x-\frac{\pi}{4})^3$, dopočítáme Taylorovy polynomy funkcí sin a cos to třetího stupně. Dostaneme:

$$
\begin{aligned}
\sin x &= \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) - \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^2}{2} - \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^3}{6} + o((x-\tfrac{\pi}{4})^3)\\
\cos x &= \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) - \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^2}{2} + \tfrac{\sqrt{2}}{2}\tfrac{(x-\frac{\pi}{4})^3}{6} + o((x-\tfrac{\pi}{4})^3)
\end{aligned}
$$

Pak už jen standardní metodou dopočítáme

$$
\lim_{x\to\frac{\pi}{4}} \left( \frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2}-\frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{6}+o((x-\frac{\pi}{4})^3)}{(1+2(x-\frac{\pi}{4})+o((x-\frac{\pi}{4}))-1)^3} \right.
$$

$$
\left. + \frac{-(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2}+\frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{6}+o((x-\frac{\pi}{4})^3))-\sqrt{2}(x-\frac{\pi}{4})}{(1+2(x-\frac{\pi}{4})+o((x-\frac{\pi}{4}))-1)^3} \right)
$$

$$
= \lim_{x\to\frac{\pi}{4}} \frac{-\sqrt{2}\frac{(x-\frac{\pi}{4})^3}{6}+o((x-\frac{\pi}{4})^3)}{8(x-\frac{\pi}{4})^3+o((x-\frac{\pi}{4})^3)} = \lim_{x\to\frac{\pi}{4}} \frac{-\frac{\sqrt{2}}{6}+\frac{o((x-\frac{\pi}{4})^3)}{(x-\frac{\pi}{4})^3}}{8+\frac{o((x-\frac{\pi}{4})^3)}{(x-\frac{\pi}{4})^3}} = -\frac{\sqrt{2}}{48}.
$$